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ON A GAME PROBLEM OF CONFLICTING CONTROL

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We consider a differential game of guidance – evasion whose solution we are required to find in the class of pure position strategies. It is shown that the introduction into this problem of information discrimination of the opponent essentially distorts the meaning of the original game problem. It is known [1–3] that a differential game of guidance–evasion has a saddle point in the class of pure position strategies if the right-hand side of the equation describing the system's dynamics satisfies the condition

$$\max_u \min_v s'f(t, x, u, v) = \min_v \max_u s'f(t, x, u, v)$$

where the maximum and minimum are computed over admissible values of u and v; s is an arbitrary n-dimensional vector, the prime denotes the transpose. However, if the stated condition is violated, then, in general, an equilibrium situation does not exist in the class of strategies. Here the game's outcome depends essentially on whether the players have information on the controls realized in the system. A typical situation is when the players do not have such information available to them; in this case an interesting problem is that of seeking the positional minimax and maximin pure strategies of the players. Below we use the results obtained in [5, 6, 9] to construct such strategies in one example of conflicting control.

1. The physical sense of the problem being investigated is the following. We have a material point moving in a horizontal plane. The motion of this point is controlled by two players who form controls which are two-dimensional vectors $u[t]$ and $v[t]$. The first player chooses the control $u[t]$, while the vector $v[t]$ is chosen by the second player, and the realizations of the controls satisfy the constraints

$$\| u[t] \| \leqslant \mu, \quad \| v[t] \| \leqslant \nu \tag{1.1}$$

Here and subsequently $\| x \|$ denotes the Euclidean norm of vector x. There is some free play in the control system. therefore, instead of the control force $w[t] = u[t] - v[t]$ a certain force $w_*[t] = u_*[t] - v_*[t]$, is applied to the point where the vectors

$u_*[t]$ and $v_*[t]$ differ from the vectors $u[t]$ and $v[t]$ by a rotation through certain angles $\alpha[t]$ and $\beta[t]$; the random errors $\alpha[t]$ and $\beta[t]$ can vary within the limits

$$|\alpha[t]| \leqslant \alpha_0 < \pi/2, \quad |\beta[t]| \leqslant \beta_0 < \pi/2 \tag{1.2}$$

We assume that the players know the present velocity and the geometric coordinates of the point. The first problem to be considered in this paper – the minimax problem – is of constructing a strategy for the first player such that for any realizations of the random errors $\alpha[t]$ and $\beta[t]$ and of control $v[t]$, it guarantees that the point can be led to an assigned position in the least possible time. The second problem – the maximin problem facing the second player – is of determining an evasion strategy, forming a control $v[t]$ such that the point does not hit onto the assigned position in a maximal time interval; here we assume that the second player does not know the random errors $\alpha[t]$, $\beta[t]$, and control $u[t]$ which are realized.

Let us pose these problems more precisely. Let $y = \{y_1, y_2\}$ be the vector made up from the geometrical coordinates of the point, $z = \{z_1, z_2\}$ be the point's velocity, $H(\gamma)$ be the transformation matrix for rotation by an angle γ, i.e.

$$H(\gamma) = \begin{Vmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{Vmatrix}$$

Then, the point's motion can be described by the equations

$$\dot{y} = z, \quad \dot{z} = H(\alpha)u - H(\beta)v \tag{1.3}$$

The game's initial position $\{t_0, x_0\}$ is specified here and subsequently $x = \{y, z\}$ is the system's four-dimensional phase vector. As the game's payoff we choose the time up to when the point $y[t]$ hits onto the origin $M = \{0, 0\}$. The first player strives to minimize the payoff, the second, to maximize it.

The multivalued functions $U = U(t, x)$ and $V = V(t, x)$, upper-semicontinuous relative to inclusion, are called the strategies of the players. Nonempty sets $U(t, x)$, $V(t, x)$ are associated with these functions of the position $\{t, x\}$ and, the elements u and v of these sets satisfy conditions (1.1). The motions of the conflict-controlled system (1.3) are defined just as in [6]. For example, every absolutely continuous vector-valued function $x[t] = x[t; t_0, x_0, U]$, $x[t_0] = x_0$, which satisfies the conditions

$$y^{\cdot}[t] = z[t], \quad z^{\cdot}[t] \in \operatorname{co}\{H(\alpha)u - H(\beta)v : \\ : u \in U(t, x[t]), |\alpha| \leqslant \alpha_0, |\beta| \leqslant \beta_0, \|v\| \leqslant \nu\}$$

for almost all $t \geqslant t_0$, is called a motion generated by a strategy $U = U(t, x)$.

In the problems being considered the players' optimal strategies are contained in the class of regularly-discontinuous strategies described above, therefore, there is no need to introduce the more complete class of positional discontinuous strategies [1, 2]. We note also that information discrimination, which is sometimes introduced to overcome difficulties of solution, would essentially distort the true meaning of the original game problems. Indeed, the assumption that the first player knows the realized random error $\alpha[t]$, would in fact eliminate the effect of this error on the control system and would lead to a false solution to the minimax problem. An analogous circumstance obtains when information discrimination is introduced in the maximin problem.

2. We consider the solution of the minimax problem. This problem consists in the construction of a strategy $U_0 = U_0(t, x)$ which possesses the following property: for any motion $x[t] = x[t; t_0, x_0, U_0]$ the condition $y[t_*] = 0$ is realized for

$t_* \leqslant t_0 + T_0$, where T_0 is some number (the minimax of the payoff), and, moreover, there does not exist a method for forming the control u, using only information on the position $\{t, x[t]\}$, which guaranteed that the point hits on to the origin $y = 0$ in a time less than T_0. In this problem the random errors $\alpha[t]$, $\beta[t]$, and the second player's control $v[t]$, all unknown to the first player and act against him. The method for forming these counteracting factors is not stipulated in the statement of the minimax problem. By the same token we do not exclude such control methods which use information on the controls $u[t]$ realized by the first player. Thus, the statement of the minimax problem facing the first player admits of an information discrimination against the first player. (It is better to overestimate the opponents capabilities than one's own).

We proceed to the construction of strategy $U_0 = U_0(t, x)$, making use of the approach proposed in [5, 6]; we introduce into consideration a hypothetical minimax mismatch $\varepsilon_0(t, x, \sigma)$, computed for the instant σ and for the initial position $\{t, x\}$. According to relation (3.9) in [6], in the case being considered

$$\varepsilon_0(t, x, \sigma) = \max_l \Big[\int_t^\sigma \min_u \max_{v, \alpha, \beta} l' \{X(\sigma, \tau) f(u, v, \alpha, \beta)\}_m \, d\tau +$$

$$+ l' \{X(\sigma, t) x\}_m\Big] = \|s_*(t, x, \sigma)\| - {}^1/_2 (\sigma - t)^2 (\mu \cos\alpha_0 - \nu)$$

$$s_* = \{y_1 + (\sigma - t) z_1, \; y_2 + (\sigma - t) z_2\} \qquad (2.1)$$

Here and subsequently l is a two-dimensional unit vector, the minimum and maximum under the integral sign are computed, respectively, over the parameter set $\{u: \|u\| \leqslant \mu\}$ and over the variable set $\{v, \alpha, \beta: \|v\| \leqslant \nu, |\alpha| \leqslant \alpha_0, |\beta| \leqslant \beta_0\}$; $X(t, \tau)$ is the 4×4 fundamental matrix of the system $y^\cdot = z$, $z^\cdot = 0$; $f(u, v, \alpha, \beta)$ is a four-dimensional vector whose first two components are zero, while the second two compose the vector $H(\alpha)u - H(\beta)v$. The subscript m in formula (2.1) equals two and signifies that we should consider the first two components of the vector occurring within the braces. Relation (2.1) is valid in the region $\{t, x, \sigma\}$ where its right-hand side is greater than zero, otherwise we set $\varepsilon_0(t, x, \sigma) = 0$. The regular case holds in the region $\varepsilon_0(t, x, \sigma) > 0$, i.e. the maximum over l in the right-hand side of expression (2.1) is attained on the unique vector $l_0(t, x, \sigma) = s_*/\|s_*\|$ and $\varepsilon_0(t, x, \sigma)$ is a continuously differentiable function of the variables t, x for a fixed value of σ.

The smallest value of the parameter $\sigma = \vartheta_0(t_0, x_0) \geqslant t_0$ for which the function $\varepsilon_0(t, x, \sigma)$ vanishes, is called the program absorption instant in minimax. The quantities $\varepsilon_0(t, x, \sigma)$, $l_0(t, x, \sigma)$, $\vartheta_0(t_0, x_0)$, introduced above, are the fundamental elements of the extremal construction defined at each position $\{t, x\}$ the values of the function $U_e = U_e(t, x)$, namely the extremal strategy which is prescribed by the following rule:

$$U_e(t, x) = \begin{cases} \mu l_0(t, x, \vartheta_0(t_0, x_0)), & \varepsilon_0(t, x, \vartheta_0(t_0, x_0)) > 0 \\ \{u: \|u\| \leqslant \mu\}, & \varepsilon_0(t, x, \vartheta_0(t_0, x_0)) = 0 \end{cases} \qquad (2.2)$$

It can be verified (for example, see the analogous case in [5, 6]) that the quantity $\varepsilon_0[t] = \varepsilon_0(t, x[t], \vartheta_0(t_0, x_0))$ does not increase along the motions $x[t] = x[t; t_0, x_0, U_e]$ generated by strategy $U_e(t, x)$; further, from the definition of the instant $\vartheta_0(t_0, x_0)$ we have the equality $\varepsilon_0[t_0] = 0$, consequently, $\varepsilon_0[\vartheta_0(t_0, x_0)] = 0$. From formula (2.1) we now obtain the equality $\varepsilon_0[\vartheta_0] = \|y[\vartheta_0]\| = 0$. Thus, the extremal strategy $U_e = U_e(t, x)$ guarantees the first player the hitting of the point $y[t]$ onto the position $M = \{0, 0\}$ at the instant ϑ_0. We now show that this result is the best one for the first player, i.e. strategy U_e is the optimal minimax strategy.

We make use of the following fact to verify this statement. Let T_* be the time of optimum response in the problem of taking the system

$$y^{\cdot} = z, \quad z^{\cdot} = w \quad (\|w\| \leqslant \mu \cos \alpha_0 - \nu) \tag{2.3}$$

from an initial state $\{t_0, x_0\}$ to the position $y = 0$. It turns out that the equality

$$T_* = \vartheta_0(t_0, x_0) - t_0 \tag{2.4}$$

is valid. This fact follows from expression (2.1) and from the corresponding solvability conditions for the time-optimal problem [7, 8].

Now suppose that the first player chooses any positional method of forming a control $u[t]$; then, setting $v[t] = u[t](\nu/\mu)$, $\beta[t] = 0$, and letting the random error $\alpha[t]$ be equal to $+\alpha_0$, $-\alpha_0$ with probability $1/2$ we get that the mean value of the norm of the vector $H(\alpha[t])\ u[t] - H(\beta[t])\ v[t]$ does not exceed the magnitude $\mu \cos \alpha_0 - \nu$, consequently [7, 8], contact cannot occur in less than the time of optimum response T_*, i.e. the estimate $T_0 \leqslant T_*$ is valid for the minimax of the payoff. On the other hand, it was established above that the strategy U_e of (2.2) ensures the hit in time $T_0 = \vartheta_0(t_0, x_0) - t_0 = T_*$. Thus, $U_e = U_e(t, x)$ is indeed the pure minimax strategy of the first player.

The class $\{U\}$ of generalized strategies, which contains the solution $U_0 = U_e$ given in (2.2) of the minimax guidance problem, was introduced for the formal description of the discontinuous control methods. Let us now clarify the significance of the result obtained in the light of the general substative interpretation given in [3, 5]. We consider the following approximation scheme for forming the first player's piecewise-constant controls $u_\Delta[t]$ $(t \geqslant t_0)$:

$$u_\Delta[t] = u_\Delta[\tau_i] \in U_e(\tau_i, x^{(\Delta)}[\tau_i]), \quad t \in [\tau_i, \tau_{i+1})$$
$$\tau_{i+1} = \tau_{i+\Delta}, \quad i = 0, 1, \ldots, \quad \tau_0 = t_0, \quad \Delta > 0$$

Here $x^{(\Delta)}[t]$ is the motion of system (1.3) generated by control $u_\Delta[t]$ and by certain realizations $\alpha[t]$, $\beta[t]$, $v[t]$. The following assertion is valid: for any arbitrarily small neighborhood S_ε of origin $M = \{0, 0\}$ we can find $\Delta_0 > 0$ such that for all $\Delta < \Delta_0$ and for any realizations $v[t]$, $\alpha[t]$, $\beta[t]$ the approximate control $u_\Delta[t]$ ensures that the point $\{y_1^{(\Delta)}[t], y_2^{(\Delta)}[t]\}$ falls into the neighborhood S_ε in time T_0. On the other hand, for any $T < T_0$ there exists a neighborhood of the origin, $S_\varepsilon(T)$, such that with a probability arbitrarily close to unity the point $\{y_1^{(\Delta)}[t], y_2^{(\Delta)}[t]\}$ will evade falling into $S_\varepsilon(T)$ on the interval $[t_0, t_0 + T]$ for Δ sufficiently small. The motion of the point $\{y_1^{(\Delta)}[t], y_2^{(\Delta)}[t]\}$ is generated by the controls $\beta[t] \equiv 0, v[t] = (\nu/\mu)\, u[t]$ (where $u[t]$ is the realization of an arbitrary positional control method of the first player), $\alpha_\Delta[t] = \alpha_\Delta[\tau_i]$ $(t \in [\tau_i, \tau_{i+1}))$ is a piecewise-constant function whose values are randomly set equal to $+\alpha_0$ and $-\alpha_0$ with probability $1/2$. It is assumed that the choice of random error $\alpha_\Delta[t]$ is probabilistically independent of the choice of the control $u[t]$.

Finally, we note the following fact. In the absence of random errors α and β the optimal payoff in the guidance problem for system (1.3) yields a quantity less than T_0; this quantity equals T_0 exactly if the first constraint in (1.1) is replaced by the condition $\|u\| \leqslant \mu \cos \alpha_0$. In other words, the appearance of the random error α has the same effect as does the lessening of that maximal force with which the first player acts on the material point being considered.

3. We now consider the maximin problem. The optimal result in the maximin problem is the quantity T^0 possessing the following property. Whatever strategy V chosen by the second player, it cannot guarantee the fulfillment of the condition $y[t] \neq 0$ for $t_0 \leqslant t \leqslant T^*$ $(T^* > T^0)$ for all motions $x[t] = x[t;\ t_0,\ x_0,\ V]$. On the other hand, strategies V_δ exist for which all motions $x[t] = x[t;\ t_0,\ x_0,\ V_\delta]$ satisfy the condition $y[t] \neq 0$ for $t_0 \leqslant t \leqslant T^0 - \delta$, here $\delta > 0$ is arbitrarily small. Thus, T^0 is the maximin of the game's payoff (more precisely, the sup inf of the payoff) in the class of pure strategies. In this paper we describe the construction of a strategy $V_\delta = V_\delta(t,\ x)$ which supplies the second player with a result as close to optimal as desired. We emphasize once again that the strategies $V_\delta = V_\delta\ (t,\ x)$ form the second player's control $v[t]$ without using information of the random error realizations $\alpha[t]$, $\beta[t]$. We have noted above that the access to such information to the second player essentially distorts the meaning of the original game problem. Furthermore, the statement of the maximin evasion problem admits of an information discrimination against that player in whose interests this problem is being solved (i.e. the second player).

To construct the strategy $V_\delta = V_\delta(t,\ x)$ we make use of the approach suggested in [9]. We introduce the concept of a hypothetical maximin mismatch $\varepsilon^0(t,x,\sigma)$ [5, 6]. Suppose that at instant t system (1.3) is found to be in state $x[t] = x$. We specify some measurable function $v[\tau]$ $(t \leqslant \tau \leqslant \sigma,\ \|v[\tau]\| \leqslant \nu)$ and we consider the attainability region $G(t,x,\sigma;v[\cdot])$, namely, the set of those points y onto which the controlled point can hit from the initial position $\{t,\ x\}$ at the instant σ when the second player chooses the stated control $v[\tau]$, $t \leqslant \tau \leqslant \sigma$, and for all possible summable realizations

$$u[\tau],\quad \alpha[\tau],\quad \beta[\tau]\quad (t \leqslant \tau \leqslant \sigma,\ \|u[\tau]\| \leqslant \mu,\ \ |\alpha[t]| \leqslant \alpha_0,\ |\beta[t]| \leqslant \beta_0).$$

This set is convex, closed and bounded. Let $\varepsilon(t,\ x,\ \sigma;\ v[\cdot])$ be the distance from point $M = \{0,\ 0\}$ to set $G(t,x,\sigma;v[\cdot])$. We now define the hypothetical maximin mismatch $\varepsilon^0(t,\ x,\ \sigma)$ as the maximal value of the quantity $\varepsilon(t,\ x,\ \sigma;\ v[\cdot])$, considered for fixed values of t, x, σ on the set of all program controls $v[\tau]$ $(t \leqslant \tau \leqslant \sigma;\ \|v[\tau]\| \leqslant \nu)$ of the second player. In the example given this definition leads to the following relation for computing $\varepsilon^0(t,x,\sigma)$:

$$\varepsilon^0(t,\ x,\ \sigma) = \max_{v[\cdot]} \varepsilon(t,\ x,\ \sigma;\ v[\cdot]) = \max_l \Big[\int_t^\sigma \max_v \min_{u,\alpha,\beta} l' \times \tag{3.1}$$

$$\times \{X(\sigma,\tau) f(u,\ v,\ \alpha,\ \beta)\}_m\, d\tau + l'\{X(\sigma,\ t)x\}_m\Big] =$$

$$= \|s_*(t,\ x,\ \sigma)\| - {}^1\!/_2(\sigma - t)^2(\mu - \nu\cos\beta_0)$$

All the quantities occurring in formula (3.1) have the same meaning as in formula (2.1), $m = 2$.

The hypothetical maximin mismatch $\varepsilon^0(t,\ x,\ \sigma)$ is completely defined by relation (3.1) in that region of the $\{t,\ x,\ \sigma\}$-space wherein the right-hand side of (3.1) is greater than zero; in the remaining part of the space we set $\varepsilon^0(t,\ x,\ \sigma) = 0$. In the region $\varepsilon^0(t,x,\sigma) > 0$ the function $\varepsilon^0(t,x,\sigma)$ is continuously differentiable in t, x for fixed σ and the maximum over $l(\|l\| = 1)$ in the right-hand side of (3.1) is achieved on the unique vector $l_0(t,\ x,\ \sigma) = s_*/\|s_*\|$, i.e. the so-called regular case takes place. The smallest roots of the equation $\varepsilon^0(t,\ x,\ \sigma) = 0$, considered for a fixed initial position $\{t,\ x\}$, is called the instant $\vartheta^0(t,\ x)$ of maximin program absorption. It turns out that the payoff's maximin T^0 in the game being considered equals $\vartheta^0 - t_0$.

where $\vartheta^0 = \vartheta^0(t_0, x_0)$.

To prove this fact we first show that none of the second player's strategies $V = V(t, x)$ guarantees that system (1.3) will evade falling into the point $M = \{0, 0\}$ on the interval $[t_0, \vartheta^0]$, i.e. $T^0 \leqslant \vartheta^0 - t_0$. We introduce into consideration the auxiliary time-optimal problem for system (2.3) from the initial position $\{t_0, x_0\}$ to the state $y = 0$, but now under the following constraints on the control resource, $\| w \| \leqslant \mu - \nu \cos \beta_0$. From equality (3.1) and from the conditions for the solvability of the time-optimal problem it follows that the time of optimum response T^* equals $\vartheta^0 - t_0$ in this case. Let us now examine the original evasion game on the interval $[t_0, \vartheta^0]$ and suppose that the second player has selected some positional method for forming the control $v[t]$. We consider the following method for forming the control u, the random errors α and β: $\alpha[t] \equiv 0$, the random error $\beta[t]$ is chosen equal to $+\beta_0$ or $-\beta_0$ with probability $1/2$, so that at each instant $t \geqslant t_0$ there is realized in system (1.3) a certain averaged value of the second player's control $v_*[t]$, not exceeding $\nu \cos \beta_0$ in absolute value. Besides this force $v_*[t]$, let there act on the material point being considered a force $u[t] = v_*[t] + w^0[t]$, where $w^0[t]$ is the solution of the auxiliary time-optimal problem. We note that the statement of the maximin problem does not exclude the possibility of the partner realizing such a control.

Thus, the force $H(\alpha)u - H(\beta)v$ acting on the point proves to be exactly equal to the vector $w^0[t]$, consequently, the motion of system (1.3), generated by these realizations, hits onto point $M = \{0, 0\}$ at the instant $t = t_0 + T^* = \vartheta^0$. Therefore, none of the second player's strategies V guarantees that the point $\{y_1[t], y_2[t]\}$ will evade falling into the state $M = \{0,0\}$ upto the instant $\vartheta^0 = t_0 + T^*$, i.e. indeed $T^0 \leqslant T^* = \vartheta^0 - t_0$.

On the other hand, we shall construct below a strategy V_δ which ensures, for all motions $x[t] = x[t; t_0, x_0, V_\delta]$ the fulfillment of the condition $y[t] \neq 0$ for $t_0 \leqslant t \leqslant \vartheta^0 - \delta$ ($\delta > 0$ is arbitrarily small). Consequently, the payoff's maximin T^0 indeed does coincide with the quantity $\vartheta^0 - t_0$.

We begin the construction of the strategy $V_\delta = V_\delta(t, x)$. As in the case of linear systems with separable controls [9], we introduce the auxiliary function

$$L_\delta(t, x) = \int_t^{\vartheta_\delta} [\varepsilon^0(t, x, \sigma)]^{-1} d\sigma, \quad \vartheta_\delta = \vartheta^0 - \delta \tag{3.2}$$

The function $L_\delta(t, x)$ is defined in that part of the $\{t, x\}$-space wherein $\varepsilon^0(t, x, \sigma) > 0$ for $t \leqslant \sigma \leqslant \vartheta_\delta$. Conversely, from the relation $L_\delta(t, x) < \infty$ we can deduce that the function $\varepsilon^0(t, x, \sigma)$ does not vanish for any $\sigma \in [t_0, \vartheta_\delta]$ (otherwise the improper integral on the right-hand side of (3.2) would diverge). In particular, $\varepsilon^0(t, x, t) = \| y[t] \| > 0$ follows from the property $L_\delta(t, x) < \infty$. Thus, it is sufficient to construct the second player's desired strategy V_δ such that along any motion $x[t] = x[t; t_0, x_0, V_\delta]$, the function $L_\delta(t, x)$ remains bounded for all $t \in [t_0, \vartheta_\delta]$. Having noted that $L_\delta(t_0, x_0) < \infty$ (since $\varepsilon^0(t, x, \sigma) > 0$ for $t_0 \leqslant \sigma \leqslant \vartheta_\delta = \vartheta^0 - \delta$), we construct the strategy $V_\delta = V_\delta(t, x)$ so as to ensure the fulfillment of the relation

$$L_\delta(t, x[t]) \leqslant L_\delta(t_0, x_0) \tag{3.3}$$

We compute the total time derivative of function $L_\delta(t, x)$ relative to the equations of motion (1.3)

$$\frac{dL}{dt} = \Phi(t, x, u, v, \alpha, \beta) = -\|y\|^{-1} - \int_t^{\vartheta_\delta} [\varepsilon^0(t, x, \sigma)]^{-2}(\sigma - t)(\mu - \nu\cos\beta_0)\, d\sigma -$$

$$-(H(\alpha)u - H(\beta)v)' \int_t^{\vartheta_\delta} [\varepsilon^0(t, x, \sigma)]^{-2}(\sigma - t)\, l_0(t, x, \sigma)\, d\sigma$$

We now compute the quantity $\Phi^0(t, x)$

$$\Phi^0(t, x) = \min_v \max_{u, \alpha, \beta} \Phi(t, x, u, v, \alpha, \beta)$$

We obtain the following relation:

$$\Phi^0(t, x) = \max_{u, \alpha, \beta} \Phi(t, x, u, v^0(t, x), \alpha, \beta) = \tag{3.4}$$

$$= -\|y\|^{-1} - (\mu - \nu\cos\beta_0)\Big(\|p(t, x)\| - \int_t^{\vartheta_\delta} (\sigma - t)[\varepsilon^0(t, x, \sigma)]^{-2} d\sigma\Big) < 0,$$

$$p(t, x) = \int_t^{\vartheta_\delta} [\varepsilon^0(t, x, \sigma)]^{-2}(\sigma - t)\, l_0(t, x, \sigma)\, d\sigma$$

$$v^0(t, x) = -\nu p(t, x) / \|p(t, x)\|$$

We now define the strategy V_δ in the following manner :

$$V_\delta = v^0(t, x) \tag{3.5}$$

Just as in [9], by using relation (3.4) we can show that strategy V_δ ensures the fulfillment of inequality (3.3) and, consequently, guarantees the fulfillment of the relation $y[t] \neq 0$ for $t_0 \leqslant t \leqslant \vartheta_\delta = \vartheta^0 - \delta$. We note that the strategy V_δ constructed in (3.5) is a continuous vector-valued function of the game's position in its domain of definition.

The significance of the probabilistic mechanism brought in to prove the optimality of the result for T^0, can be clarified once again by the general interpretation given in [3]. Suppose that the random error β is formed at discrete instants $t = \tau_i$ ($\tau_i = t_0 + i\Delta$, $\Delta > 0$) and takes value $+\beta_0$ or $-\beta_0$ with probability $1/2$. Next we set $\beta[t] = \beta[\tau_i]$ for $t \in [\tau_i, \tau_{i+1})$; $\alpha[t] \equiv 0$, $u[t] = u^0[t] + v[t]$, where the choice of the values of the random error $\beta[t]$, realized in system (1.3) at a given instant, is probabilistically independent of the choice of the control $v[t]$. With probability arbitrarily close to unity the motions generated by these realizations fall into an arbitrarily small neighborhood of the point $M = \{0, 0\}$ not later than the instant ϑ^0 when the subdivision $\Delta > 0$ is chosen sufficiently fine.

In conclusion, we note that the payoff's minimax T_0 coincides with the value of the differential game described by the equation

$$y^{\cdot} = z, \qquad z^{\cdot} = u - v \tag{3.6}$$

under the constraints on the controls $\|u\| \leqslant \mu\cos\alpha_0$ and $\|v\| \leqslant \nu$. The payoff's maximin T^0 is the value of the differential game described by system (3.6) under the constraints on the controls $\|u\| \leqslant \mu$ and $\|v\| \leqslant \nu\cos\beta_0$. Thus, the presence of random error leads to the same result as does the lessening of the resources of the player from whose point of view the game problem is being investigated.

In proving the optimality of the pure strategies U_ε and V_δ constructed above, we considered the mixed strategies of the opponent, namely, in the minimax problem we introduced a mixed control by the random error α, while in the maximin problem, a mixed control by the random error β. These notions were brought in for auxiliary arguments. However, if we pose the problem of seeking the optimal mixed strategies in a differential game corresponding to Esq. (1.3), where the parameters u and β are subject to the first player, while v and α. to the second player, then we can show that this pursuit–evasion game has a value T_c in the class of mixed strategies. The quantity T_c coincides with the value of the differential game described by Eqs. (3.6) under the constraints $\|u\| \leqslant \mu \cos \alpha_0$ and $\|v\| \leqslant \nu \cos \beta_0$. The problem statements and the construction of the optimal strategies, presented in this paper, can be carried over to the case when instead of system (1.3) we consider a check model system [10] in which, however, random errors of the "free-play" type also are imposed on the players' controls.

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